

POINTWISE CHARACTERISTIC FACTORS FOR WIENER-WINTNER DOUBLE RECURRENCE THEOREM

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ABSTRACT. In this paper, we extend Bourgain's double recurrence result to the Wiener-Wintner averages. Let (X, \mathcal{F}, μ, T) be a standard ergodic system. We will show that for any $f_1, f_2 \in L^\infty(X)$, the double recurrence Wiener-Wintner average

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t}$$

converges off a single null set of X independent of t as $N \rightarrow \infty$. Furthermore, we will show a uniform Wiener-Wintner double recurrence result: If either f_1 or f_2 belongs to the orthogonal complement of the Conze-Lesigne factor, then there exists a set of full measure such that the supremum on t of the absolute value of the averages above converges to 0.

1. HISTORICAL BACKGROUND

In 1990, Bourgain proved the result on double recurrence [7], which is stated as follows:

Theorem 1.1 (Bourgain [7]). *Let (X, \mathcal{F}, μ, T) be an ergodic system, and T_1, T_2 be powers of T . Then, for $f_1, f_2 \in L^\infty(\mu)$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_1(T_1^n x) f_2(T_2^n x)$$

converges for μ -a.e. $x \in X$.

In [7], the theorem was proven for the case $T_1 = T = T_2^{-1}$. Bourgain's proof relies on the uniform Wiener Wintner theorem, which is stated as follows (see, for example, [2] for a proof):

Theorem 1.2. *Let (X, \mathcal{F}, μ, T) be an ergodic system, and let f be a function in the orthogonal complement of the Kronecker factor of (X, T) . Then there exists a set of full measure X_f such that*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n t} \right| = 0$$

for all $x \in X_f$.

In 2001, the second author worked on an extended result of Bourgain in his Ph. D. thesis [10], and proved the double recurrence Wiener Wintner result for the case when T is totally ergodic (i.e. T^a is ergodic for any $a \in \mathbb{Z}$).

Theorem 1.3. *Let (X, \mathcal{F}, μ, T) be a standard ergodic dynamical system (i.e. X is a compact metrizable space, \mathcal{F} is a Borelian sigma-algebra, μ is a probability Borel measure, and T is a self-homeomorphism). Let $f_1, f_2 \in L^2(X)$. Let \mathcal{CL} be the maximal isometric extension of the Kronecker factor of T . Let*

$$W_N(f_1, f_2, x, t) = \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t}$$

- (1) (Double Uniform Wiener-Wintner Theorem) *If either f_1 or f_2 belongs to \mathcal{CL}^\perp , then there exists a set of full measure $X_{f_1 \otimes f_2}$ such that for all $x \in X_{f_1 \otimes f_2}$,*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} |W_N(f_1, f_2, x, t)| = 0$$

- (2) (General Convergence) If $f_1, f_2 \in \mathcal{CL}$, then $W_N(f_1, f_2, x, t)$ converges for μ -a.e. $x \in X$ for all $t \in \mathbb{R}$, provided that the cocycle associated with \mathcal{CL} is affine.

Theorem 1.3 was proved in several stages. For (1), first one identifies the pointwise limit of the double recurrence averages as an integral with respect to a particular Borel measure (disintegration). Then one uses Wiener's lemma on the continuity of spectral measures and van der Corput's inequality to show that the double recurrence average converges to 0. For the second part, one first shows that the total ergodicity of T asserts that \mathcal{CL} for every integer power of T are the same, which allows one to assume that both functions lie in the same factor of $L^2(X, \mu)$. Furthermore, the assumption that the measurable cocycle associated with \mathcal{CL} is affine allows one to use the homomorphism property to simplify the computations.

A little was known about characteristic factors back then, especially for pointwise convergence. Originally in [10], the factor \mathcal{CL} is referred to as "Conze-Lesigne" factor, as they first appeared in series of work by Conze and Lesigne (see, for example, [8, 9] for details). But the definition of Conze-Lesigne factor has been updated since then, particularly since the emergence of Host-Kra-Ziegler factors in [14]. In [6], it is noted that the updated Conze-Lesigne factor \mathcal{Z}_2 is smaller than \mathcal{CL} , so more work is needed to prove the uniform Wiener-Wintner theorem for the case either $f_1, f_2 \in \mathcal{Z}_2^\perp$ since $\mathcal{CL}^\perp \subset \mathcal{Z}_2^\perp$.

2. INTRODUCTION

In this paper, we will prove the uniform Wiener-Wintner result for the case $f_1 \in \mathcal{Z}_2^\perp$ using the seminorms that characterize these related factors. These characteristic factors and seminorms were developed in the work of Host and Kra [14] and Ziegler [19] independently. They are similar to the seminorms introduced by Gowers in [13].

Definition 2.1. Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system on a probability measure space. The factors \mathcal{Z}_k are defined in terms of seminorms as follows.

- The factor \mathcal{Z}_0 is the trivial σ -algebra.
- The factor \mathcal{Z}_1 can be characterized by the seminorm $||| \cdot |||_2$ where

$$|||f|||_2^4 = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \left| \int f \cdot f \circ T^h d\mu \right|^2,$$

i.e. A function $f \in \mathcal{Z}_1^\perp$ if and only if $|||f|||_2 = 0$.

- The factor \mathcal{Z}_2 is the Conze-Lesigne factor. Functions in this factor are characterized by the seminorm $||| \cdot |||_3$ such that

$$|||f|||_3^8 = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H |||f \cdot f \circ T^h|||_2^4,$$

i.e. A function $f \in \mathcal{Z}_2^\perp$ if and only if $|||f|||_3 = 0$.

- More generally, B. Host and B. Kra showed in [14] that for each positive integer k , we have

$$|||f|||_{k+1}^{2^{k+1}} = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H |||f \cdot f \circ T^h|||_k^{2^k}$$

with the condition that $f \in \mathcal{Z}_{k-1}^\perp$ if and only if $|||f|||_k = 0$.

In 2012, Assani and Presser published an update [6] of their earlier unpublished work [5] on characteristic factors and the Multiterm Return Times Theorem.

Definition 2.2. Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system on a probability measure space. We define factors \mathcal{A}_k in the following inductive way.

- The factor \mathcal{A}_0 is the trivial σ -algebra $\{X, \emptyset\}$.
- The factor \mathcal{A}_1 is the Kronecker factor of T . We denote $N_1(f) = \|\mathbb{E}(f|\mathcal{A}_1)\|_2$.
- For $k \geq 1$, the factor \mathcal{A}_{k+1} is characterized by the following: A function $f \in \mathcal{A}_{k+1}^\perp$ if and only if

$$N_{k+1}(f)^4 := \lim_H \frac{1}{H} \sum_{h=1}^H \|\mathbb{E}(f \cdot f \circ T^h | \mathcal{A}_k)\|_2^2 = 0.$$

It was proven that the quantities $N_k(f)$ are well-defined in [1], and they characterize factors \mathcal{A}_k of T which are successive maximal isometric extensions. These successive factors turned out to be the k -step distal factors introduced by H. Furstenberg [11].

Furthermore, in [6], it was shown that for any measure-preserving system (Y, \mathcal{G}, ν, S) , we have

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(S^n y) e^{2\pi i n t} \right|^2 \leq C N_3(f_1)^2.$$

It is known that $\mathcal{Z}_k \subset \mathcal{A}_k$ (where $\mathcal{Z}_0 = \mathcal{A}_0$, and $\mathcal{Z}_1 = \mathcal{A}_1$), so $\mathcal{Z}_k^\perp \supset \mathcal{A}_k^\perp$. In [6], it was proven that \mathcal{Z}_k are pointwise characteristic for the multiterm return times averages.

In this paper, we will update Theorem 1.3 in the following ways:

- We will only assume that T is ergodic, rather than totally ergodic.
- We will show that \mathcal{Z}_2 (and \mathcal{A}_2) is a characteristic factor for this Wiener-Wintner average, i.e. We will prove the uniform double Wiener-Wintner result for the case either $f_1, f_2 \in \mathcal{Z}_2^\perp$ rather than $\mathcal{C}\mathcal{L}^\perp$.
- We will show that the convergence holds in general for case $f_1, f_2 \in \mathcal{Z}_2$.

In other words, we will prove the following:

Theorem 2.3. *Let (X, \mathcal{F}, μ, T) be a standard ergodic dynamical system. Let $f_1, f_2 \in L^2(X)$. Let*

$$W_N(f_1, f_2, x, t) = \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an} x) f_2(T^{bn} x) e^{2\pi i n t}.$$

- (1) (Double Uniform Wiener-Wintner Theorem) *If either f_1 or f_2 belongs to \mathcal{Z}_2^\perp , then there exists a set of full measure X_{f_1, f_2} such that for all $x \in X_{f_1, f_2}$,*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} |W_N(f_1, f_2, x, t)| = 0$$

- (2) (General Convergence) *If $f_1, f_2 \in \mathcal{Z}_2$, then $W_N(f_1, f_2, x, t)$ converges for μ -a.e. $x \in X$ for all $t \in \mathbb{R}$.*

We will use the seminorms mentioned above to prove (1). We will first show that (1) holds for the case $f_1 \in \mathcal{Z}_1$ when $a = 1$ and $b = 2$ in section 3. Complication arises when $|b - a| > 1$, and we will prove the case for general $a, b \in \mathbb{Z}$ in section 4. In section 5, we will prove the uniform double Wiener-Wintner result for the case f_1 belongs to \mathcal{A}_2^\perp , and $a = 1$ and $b = 2$; while this is the special case of the preceding result, we will note that the seminorms for \mathcal{A}_k gives pointwise estimate of the average rather than the norm estimate that we obtained by using \mathcal{Z}_k seminorms. Finally, in section 6, we will show (2) of Theorem 2.3 using Leibman's convergence result in [16].

Throughout this paper, we will assume that the functions f_1 and f_2 are real-valued without loss of generality, unless specified otherwise.

3. WHEN $f_1 \in \mathcal{Z}_2^\perp$, $a = 1$, $b = 2$

In this section, we will prove the uniform Wiener Wintner theorem for the case $f_1 \in \mathcal{Z}_2^\perp$. We will prove this special case since the fact that $|b - a| = 1$ simplifies the proofs tremendously, because $f, g \in L^2(\mu)$,

$$\int f(Tx) g(T^2 x) d\mu(x) = \int f(x) g(Tx) d\mu(x).$$

We will present two proofs; the first one is more direct and concise than the second one, while the second one will be similar to the proof for the general case (when $a, b \in \mathbb{Z}$).

Theorem 3.1. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system, and $f_1, f_2 \in L^\infty(X)$, and $\|f_2\|_\infty = 1$. If $f_1 \in \mathcal{Z}_2^\perp$, then*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right| = 0$$

for μ -a.e. $x \in X$.

First proof. We denote $F_{1,h}(x) = (f_1 \cdot f_1 \circ T^h)(x)$ and $F_{2,h}(x) = (f_2 \cdot f_2 \circ T^{2h})(x)$. We first note that, by van der Corput's lemma, for any $0 < H < N$, we have

$$\begin{aligned} & \int \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) \right|^2 d\mu \\ & \leq \frac{C}{H} + \frac{2}{(H+1)^2} \sum_{h=1}^H (H-h+1) \left(\frac{1}{N} \sum_{n=1}^{N-h} \int F_{1,h}(T^n x) F_{2,h}(T^{2n} x) d\mu \right) \\ & = \frac{C}{H} + \frac{2}{H} \sum_{h=1}^H \left(\int F_{1,h}(x) \frac{1}{N} \sum_{n=1}^{N-h} F_{2,h}(T^n x) d\mu \right) \end{aligned}$$

By Birkhoff's pointwise ergodic theorem, we would have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-h} F_{2,h}(T^n x) \\ & = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N F_{2,h}(T^n x) - \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=N-h+1}^N F_{2,h}(T^n x) \\ & = \int F_{2,h}(y) d\mu(y) \end{aligned}$$

since

$$\lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=N-h+1}^N F_{2,h}(T^n x) \right| \leq \lim_{N \rightarrow \infty} \frac{h}{N} \|f_2\|_\infty^2 = 0.$$

Therefore,

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \int \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) \right|^2 d\mu \\ & \leq \frac{1}{H} \sum_{h=1}^H \left| \int F_{1,h}(x) d\mu(x) \int F_{2,h}(y) d\mu(y) \right| \\ (1) \quad & \leq \frac{\|f_2\|_\infty}{H} \sum_{h=1}^H \left| \int F_{1,h}(x) d\mu(x) \right| \leq C \|f_1\|_2^2. \end{aligned}$$

Now we are ready to prove the uniform Wiener Wintner result. Again, by van der Corput's lemma, we obtain

$$\begin{aligned} & \int \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 d\mu \\ & \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \int \left| \frac{1}{N} \sum_{n=1}^{N-h} f_1(T^n x) f_1(T^{n+h} x) f_2(T^{2n} x) f_2(T^{2n+2h} x) \right| d\mu \end{aligned}$$

By Cauchy-Schwarz inequality and (1),

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N-h} f_1(T^n x) f_1(T^{n+h} x) f_2(T^{2n} x) f_2(T^{2n+2h} x) \right| d\mu \\ & \leq \left(\limsup_{N \rightarrow \infty} \int \left| \frac{1}{N} \sum_{n=1}^{N-h} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2 d\mu \right)^{1/2} \\ & \leq C \|f_1 \cdot f_1 \circ T^h\|_2^2 \end{aligned}$$

Since

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|f_1 \cdot f_1 \circ T^h\|_2^2 \leq \left(\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|f_1 \cdot f_1 \circ T^h\|_2^4 \right)^{1/2} = \|f_1\|_3^4,$$

we must have

$$\int \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^{N-h} f_1(T^n x) f_1(T^{n+h} x) f_2(T^{2n} x) f_2(T^{2n+2h} x) \right| d\mu \leq C \|f_1\|_3^4$$

since we can let $H \rightarrow \infty$ as $N \rightarrow \infty$. Since $\sup_{t \in \mathbb{R}} \frac{1}{N} \sum_{n=1}^N |f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t}| \geq 0$ and $\|f_1\|_3 = 0$, we can conclude that

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right| = 0$$

for μ -a.e. $x \in X$. □

Second Proof. We denote $F_{1,h}(x) = f_1(x) f_1 \circ T^h(x)$, and $F_{2,h}(x) = f_2(x) f_2 \circ T^{2h}(x)$. By van der Corput's lemma, we have, for any $0 < H < N$,

$$\begin{aligned} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 &\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right| \\ &\leq \frac{C}{H} + \frac{C}{H} \left(\sum_{h=1}^H \left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2 \right)^{1/2}, \end{aligned}$$

where the second inequality is the consequence of the Cauchy-Schwarz inequality. Note that we can again apply the van der Corput's lemma on the average $\left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2$ to obtain the following bound for $0 < K < N$.

$$\begin{aligned} &\left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2 \\ &\leq \frac{C'}{K} + \frac{C'}{K} \sum_{k=1}^K \frac{1}{N} \sum_{n=1}^{N-k-1} (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x) (F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x). \end{aligned}$$

Note that the average

$$\frac{1}{N} \sum_{n=1}^{N-k-1} (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x) (F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x)$$

converges by the Bourgain's a.e. double recurrence theorem in [7]. Therefore, by the dominated convergence theorem,

$$\int \lim_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right| d\mu(x) = \lim_{N \rightarrow \infty} \int \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right| d\mu(x)$$

Also, by Hölder's inequality,

$$\begin{aligned}
& \int \left(\frac{1}{H} \sum_{h=1}^H \left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2 \right)^{1/2} \\
& \leq \left(\frac{1}{H} \sum_{h=1}^H \int \left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^n x) F_{2,h}(T^{2n} x) \right|^2 \right)^{1/2} \\
& \leq \left(\frac{1}{H} \sum_{h=1}^H \left(\frac{C'}{K} + \frac{C'}{K} \sum_{k=1}^K \left(\frac{1}{N} \sum_{n=1}^{N-k-1} \int (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x) (F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x) \right) \right) \right)^{1/2}
\end{aligned}$$

And note that, by Birkhoff's pointwise ergodic theorem and the dominated convergence theorem, we have

$$\begin{aligned}
& \frac{1}{N} \sum_{n=1}^{N-k-1} \int (F_{1,h} \cdot F_{1,h} \circ T^k)(T^n x) (F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^{2n} x) d\mu(x) \\
& = \int (F_{1,h} \cdot F_{1,h} \circ T^k)(x) \left(\frac{1}{N} \sum_{n=1}^{N-k-1} (F_{2,h} \cdot F_{2,h} \circ T^{2k})(T^n x) \right) d\mu(x) \\
& \xrightarrow{N \rightarrow \infty} \int \int F_{1,h}(x) (F_{1,h} \circ T^k)(x) F_{2,h}(y) (F_{2,h} \circ T^{2k})(y) d\mu(x) d\mu(y) \\
& = \iint f_1 \otimes f_2(x, y) f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y)) d\mu \otimes \mu(x, y),
\end{aligned}$$

where $U = T \otimes T^2$ is a measure preserving transformation on X^2 . If we take ergodic decomposition of $\mu \otimes \mu$ with respect to U , then the integral becomes

$$\iint f_1 \otimes f_2(x, y) f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y)) d(\mu \otimes \mu)_c(x, y) d\mu(c)$$

Let $H = K$. Note that, on the system $(X^2, (\mu \otimes \mu)_c, U)$ for a.e. $c \in X$, Lemma 5 from [3] tells us that we have

$$\begin{aligned}
& \left| \frac{1}{H^2} \sum_{h,k=0}^{H-1} f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y)) \right|^2 \\
(2) \quad & \leq \limsup_{H \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{H} \sum_{h=1}^H f_1 \otimes f_2(U^h(x, y)) e^{2\pi i h t} \right|^2.
\end{aligned}$$

So if we can show that $f_1 \in \mathcal{Z}_2^\perp$ implies that $f_1 \otimes f_2$ belongs to the orthogonal complement of the Kronecker factor with respect to the transformation U and measure $(\mu \otimes \mu)_c$ for μ -a.e. $c \in X$, we can show that the average converges to 0.

To show this property, we first observe the following lemma:

Lemma 3.2. *Suppose (Y, \mathcal{G}, ν, U) is a measure preserving system, Take $f \in L^\infty(X)$. Assume*

$$\int \limsup_N \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f(U^n y) e^{2\pi i n t} \right| d\nu = 0.$$

If σ_f is the spectral measure of f with respect to U , then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\hat{\sigma}_f(n)|^2 = 0.$$

Proof. By Wiener's theorem, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\hat{\sigma}_f(n)|^2 = \sum_t |\sigma_f(\{t\})|^2$$

And observe that, by the spectral theorem,

$$|\sigma_f(\{t\})| = \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \hat{\sigma}_f(n) e^{2\pi i n t} \right| = \left| \lim_{N \rightarrow \infty} \int f(y) \frac{1}{N} \sum_{n=1}^N f(U^n y) e^{2\pi i n t} d\nu(y) \right|.$$

Since $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(U^n y) e^{2\pi i n t}$ converges by Wiener Wintner pointwise ergodic theorem, we can apply Lebesgue's dominated convergence theorem to show that

$$\begin{aligned} |\sigma_f(\{t\})| &= \left| \int f(y) \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(U^n y) e^{2\pi i n t} d\nu(y) \right| \\ &\leq \|f\|_\infty \int \limsup_N \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f(U^n y) e^{2\pi i n t} \right| d\nu(y) = 0. \end{aligned}$$

□

Lemma 3.3. Suppose $f_1 \in \mathcal{Z}_2^\perp$. Then $f_1 \otimes f_2$ belongs to the orthogonal complement of the Kronecker factor of $U = T \times T^2$ with respect to measure $\mu \otimes \mu_c$ for μ -a.e. c .

Remark: A similar result was proven by Rudolph in [18]. In his work, Conze-Lesigne algebra referred is the maximal isometric extension of the Kronecker factor, which is \mathcal{CL} in Theorem 1.3.

Proof. Equivalently, we would like to show that if $\sigma_{f_1 \otimes f_2}$ is the spectral measure with respect to U , we have

$$\frac{1}{N} \sum_{n=1}^N |\hat{\sigma}_{f_1 \otimes f_2}(n)|^2 \rightarrow 0$$

i.e. we would like to show that $\sigma_{f_1 \otimes f_2}$ is a continuous measure. By lemma 3.2, this can be done by showing

$$(3) \quad \int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1 \otimes f_2(U^n(x, y)) e^{2\pi i n t} \right| d(\mu \otimes \mu)_c = 0$$

for μ -a.e. $c \in X$. Note that

$$\begin{aligned} &\iint \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1 \otimes f_2(U^n(x, y)) e^{2\pi i n t} \right| d\mu \otimes \mu_c d\mu(c) \\ &= \int \left(\int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} y) e^{2\pi i n t} \right| d\mu(y) \right) d\mu(x) \end{aligned}$$

Note that, by Lemma 8 in [6], we know that

$$\int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} y) e^{2\pi i n t} \right| d\mu(y) \leq C \|f_1\|_3^2 = 0.$$

Therefore,

$$(4) \quad \iint \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1 \otimes f_2(U^n(x, y)) e^{2\pi i n t} \right| d(\mu \otimes \mu)_c d\mu(c) = 0.$$

Since

$$\int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1 \otimes f_2(U^n(x, y)) e^{2\pi i n t} \right| d(\mu \otimes \mu)_c$$

is a.e. non-negative function of c that is measurable with respect to μ , we can deduce that (4) implies that (3) equals 0. □

Because of lemma 3.3, we can show that the average (2) converges to 0, which proves theorem 3.1. □

4. CASE $f_1 \in \mathcal{Z}_2^\perp$, WHEN $a, b \in \mathbb{Z}$.

Here, we will prove the uniform Wiener Wintner double recurrence property for any $a, b \in \mathbb{Z}$. We will first prove various lemmas to overcome the obstacle caused by the fact that T^a is no longer ergodic.

The following lemma will show that for any T^a , any T^a -invariant function f can be expressed in terms of an integral kernel (that does not depend on f). The kernel first appeared in Theorem 2.1 of [12]; we will present a detailed proof here. This kernel will be useful to characterize various conditional expectations.

Lemma 4.1. *Let T be an ergodic map, and s be a positive integer. Then there exist a disjoint partition of T^s -invariant sets A_1, \dots, A_l such that every T^s -invariant function f can be expressed as an integral with respect to the kernel*

$$(5) \quad K(x, y) = l \sum_{k=1}^l \mathbb{1}_{A_k}(x) \mathbb{1}_{A_k}(y).$$

Proof. If T^s is ergodic, we are done, since f is a constant. If not, suppose A is a T^s -invariant subset of X such that $0 < \mu(A) < 1$. Define a function

$$f_A := \mathbb{1}_A + \mathbb{1}_{T^{-1}A} + \mathbb{1}_{T^{-2}A} + \dots + \mathbb{1}_{T^{-(s-1)}A}$$

Observe that f_A is T -invariant, and since T is ergodic, f_A must be a constant. Therefore,

$$\mathbb{1}_A + \mathbb{1}_{T^{-1}A} + \mathbb{1}_{T^{-2}A} + \dots + \mathbb{1}_{T^{-(s-1)}A} = \int f_A d\mu = s\mu(A).$$

Note that $f_A \neq 0$, since $\mu(A) \neq 0$. Similarly, $f_A \neq s$, since $\mu(A) \neq 1$. If $f_A = 1$, then for μ -a.e. $x \in X$, $\mathbb{1}_A + \mathbb{1}_{T^{-1}A} + \mathbb{1}_{T^{-2}A} + \dots + \mathbb{1}_{T^{-(s-1)}A} = 1$, which implies that $\mu(T^{-i}A \cap T^{-j}A) = 0$ for any $0 \leq i < j \leq s-1$. Hence, $A, T^{-1}A, \dots, T^{-(s-1)}A$ are disjoint, and furthermore, $\mu(X) = \sum_{k=0}^{s-1} \mu(T^{-k}A) = 1$, so $A, T^{-1}A, \dots, T^{-(s-1)}A$ is a partition of X .

Now we show that A (and similarly, $T^{-1}A, \dots, T^{-(s-1)}A$) is an atom (of a collection of T^s -invariant sets). If $B \subset A$ and B is T^s -invariant, then

$$f_B = \mathbb{1}_B + \mathbb{1}_{T^{-1}B} + \mathbb{1}_{T^{-2}B} + \dots + \mathbb{1}_{T^{-(s-1)}B} = s\mu(B) \leq s\mu(A) = 1$$

The above holds only when $\mu(B) = 0$ or $\mu(B) = 1/s = \mu(A)$, which implies that $B = A$ μ -a.e. For $k > 0$, we note that if $B \subset T^{-k}A$ is T^s -invariant, then $T^k B \subset A$ is also T^s -invariant, so if $\mu(B) \neq 0$, then $\mu(B) = \mu(T^k B) = \mu(A) = \mu(T^{-k}A)$, which proves that $T^{-k}A$ is also an atom for $k > 0$.

If f is a T^s -invariant function, then we claim that

$$(6) \quad f = \sum_{k=0}^{s-1} \left(\frac{\int_{T^{-k}A} f d\mu}{\mu(T^{-k}A)} \right) \mathbb{1}_{T^{-k}A} = s \sum_{k=0}^{s-1} \left(\int_{T^{-k}A} f d\mu \right) \mathbb{1}_{T^{-k}A}.$$

First, we note that the \mathcal{S} , the σ -algebra generated by the sets $A, T^{-1}A, \dots, T^{-(s-1)}A$, is a collection of finite union of sets $A, T^{-1}A, \dots, T^{-(s-1)}A$. Next, we know that f is \mathcal{S} -measurable. Indeed

$$\{f > \lambda\} = \bigcup_{k=0}^{s-1} \left(\{f > \lambda\} \cap T^{-k}A \right),$$

and we note that $\{f > \lambda\} \cap T^{-k}A$ is T^s -invariant. Since $T^{-k}A$ is an atom for each k , we must have $\{f > \lambda\} \cap T^{-k}A = T^{-k}A$ or an empty set. This implies that $\{f > \lambda\} \in \mathcal{S}$.

Since we know that f is \mathcal{S} -measurable, we note that f can be expressed as the expression above (a fact regarding conditional expectation). This proves the claim. Since (6) holds, if we denote $T^{-k}A = A_k$, then we have

$$f \circ T^s(x) = f(x) = \int s \sum_{k=0}^{s-1} \mathbb{1}_{A_k}(y) \mathbb{1}_{A_k}(x) f(y) d\mu(y) = \int f(y) K(x, y) d\mu(y)$$

which proves the lemma for the case $f_A = 1$.

Now, suppose $f_A = k$ for $2 \leq k \leq s-1$. Let $B = T^{-l_1}A \cap T^{-l_2}A \cap \dots \cap T^{-l_k}A$, where $0 \leq l_1 < l_2 < \dots < l_k \leq s-1$, and $\mu(B) > 0$ (we know such B exists since $f_A = k$). Define

$$f_B = \mathbb{1}_B + \mathbb{1}_{T^{-1}B} + \dots + \mathbb{1}_{T^{-(s-1)}B}$$

Note that f_B is T -invariant, so it must be a constant that equals to $s\mu(B)$. Since $\mu(B) > 0$, we know that $f_B > 0$.

Also, note that each $T^{-j}B$ is disjoint for $0 \leq j \leq s-1$. Assume it is not. Then for some $0 \leq i < j \leq s-1$, there exists $x \in T^{-i}B \cap T^{-j}B$ such that $f_A(x) > k$, which is a contradiction. Therefore, we must have $f_B \leq 1$, and we can conclude that $f_B = 1$. By letting $A_i = T^{-i}B$, we have proved the lemma. \square

The next lemma will provide a simple yet useful comparison between $\limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|f \cdot f \circ T^{ah}\|_k^{2k}$ and $\|f\|_{k+1}^{2k+1}$.

Lemma 4.2. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system, and $a \in \mathbb{Z}$. Then for any positive integer k , we have*

$$\limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|f \cdot f \circ T^{ah}\|_k^{2k} \leq |a| \|f\|_{k+1}^{2k+1}.$$

Proof. Note that

$$\frac{1}{H} \sum_{h=1}^H \|f \cdot f \circ T^{ah}\|_k^{2k} \leq \frac{1}{H} \sum_{h=1}^{|a|H} \|f \cdot f \circ T^h\|_k^{2k} = |a| \left(\frac{1}{|a|H} \sum_{h=1}^{|a|H} \|f \cdot f \circ T^h\|_k^{2k} \right).$$

The sequence $\left(\frac{1}{|a|H} \sum_{h=1}^{|a|H} \|f \cdot f \circ T^h\|_k^{2k} \right)_H$ is a subsequence of $\left(\frac{1}{H} \sum_{h=1}^H \|f \cdot f \circ T^h\|_k^{2k} \right)_H$, which converges to $\|f\|_{k+1}^{2k+1}$. Hence,

$$\limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|f \cdot f \circ T^{ah}\|_k^{2k} \leq |a| \left(\lim_{H \rightarrow \infty} \frac{1}{|a|H} \sum_{h=1}^{|a|H} \|f \cdot f \circ T^h\|_k^{2k} \right) = |a| \|f\|_{k+1}^{2k+1}$$

\square

The proof of Bourgain's double recurrence theorem [7] relies on the classical uniform Wiener-Wintner theorem, which holds for the case when T is ergodic. Here, we prove the uniform Wiener Wintner theorem that holds for the case when the measure preserving transformation is a power of ergodic map. This allows us to use Bourgain's double recurrence theorem without assuming total ergodicity.

Theorem 4.3. *Let (X, \mathcal{F}, μ, T) be an ergodic system. Suppose $f \in \mathcal{Z}_1^\perp$. There exists a set of full measure X_f such that for any $x \in X_f$ and for any integer a , we have*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f(T^{an}x) e^{2\pi i n t} \right| = 0$$

Proof. We denote $F_h(x) = f(x) \cdot f \circ T^{ah}(x)$. We first apply van der Corput's lemma to obtain the following:

$$\left| \frac{1}{N} \sum_{n=1}^N f(T^{an}x) e^{2\pi i n t} \right|^2 \leq \frac{C}{H} + \frac{C}{H^2} \sum_{h=1}^H (H+1-h) \Re \left(e^{-2\pi i h t} \frac{1}{N} \sum_{n=1}^{N-h} F_h(T^{an}x) \right)$$

Since for each h ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-h} F_h(T^{an}x) = \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=1}^N F_h(T^{an}x) - \frac{1}{N} \sum_{n=N-h+1}^N F_h(T^{an}x) \right) = \mathbb{E}_a(f \cdot f \circ T^{ah})(x)$$

where \mathbb{E}_a is the conditional expectation operator to the sigma-algebra of T^a -invariant sets. Using the integral kernel from lemma 4.1, we obtain

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N f(T^{an}x) e^{2\pi i n t} \right|^2 \\ & \leq \frac{C}{H} + \frac{C}{H^2} \sum_{h=1}^H (H+1-h) \Re \left(e^{-2\pi i h t} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-h} F_h(T^{an}x) \right) \\ & = \frac{C}{H} + \frac{C}{H^2} \sum_{h=1}^H (H+1-h) \Re \left(e^{-2\pi i h t} \mathbb{E}_a(f \cdot f \circ T^{ah})(x) \right) \\ & \leq \Re \left(\frac{C}{H} + \frac{C}{H} \sum_{h=1}^H e^{-2\pi i h t} \int \mathbb{1}_{A_i}(y) f(y) f(T^{ah}y) d\mu(y) \right) \end{aligned}$$

where A_i is one of the sets of the partition given in lemma 4.1 such that $x \in A_i$. Set $g(y) = \mathbb{1}_{A_i}(y) f(y)$. Then we notice that

$$\int g(y) f(T^{ah}y) d\mu(y) = \hat{\sigma}_{f,g,T^a}(h),$$

where σ_{f,g,T^a} is the spectral measure of functions f and g with respect to the transformation T^a . Note that σ_{f,g,T^a} is absolutely continuous with respect to σ_{f,T^a} (see, for example, Proposition 2.4 of [17] for a proof), and we note that σ_{f,T^a} is a continuous measure since $f \in \mathcal{Z}_1^\perp$ and $\sigma_{f,T}$ is a continuous measure, so by

Wiener's theorem, $\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H |\hat{\sigma}_{f,T}(h)|^2 = 0$. Hence,

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H |\hat{\sigma}_{f,T^a}(h)|^2 = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H |\hat{\sigma}_{f,T}(ah)|^2 \leq \lim_{H \rightarrow \infty} |a| \left(\frac{1}{|a|H} \sum_{h=1}^{|a|H} |\hat{\sigma}_{f,T}(h)|^2 \right) = 0,$$

and again by Wiener's theorem, σ_{f,T^a} is a continuous measure. Hence, σ_{f,g,T^a} is continuous. Therefore,

$$(7) \quad 0 = \sigma_{f,g,T^a}(\{t\}) = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H e^{-2\pi i h t} \int g(y) f(T^{ah}y) d\mu(y),$$

as desired.

To show that the uniform convergence holds, we note that

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f(T^{an}x) e^{2\pi i n t} \right|^2 & \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-h} F_h(T^{an}x) \right| \\ & = \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \left| \mathbb{E}_a(f \cdot f \circ T^{ah})(x) \right| \end{aligned}$$

Let γ_x be a measure on \mathbb{R} such that $\hat{\gamma}_x(h) = \mathbb{E}_a(f \cdot f \circ T^{ah})(x)$. By (7), we know that for all $t \in \mathbb{R}$,

$$\gamma_x(\{-t\}) = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \hat{\gamma}_x(h) e^{-2\pi i h t} = 0$$

so γ_x is a continuous measure. Therefore, by Wiener's theorem, we would obtain

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f(T^{an}x) e^{2\pi i n t} \right|^2 \leq \lim_{H \rightarrow \infty} \left(\frac{C}{H} + \frac{C}{H} \sum_{h=1}^H |\hat{\gamma}_x(h)| \right) = 0$$

□

Here we introduce seminorms that are similar the ones introduced in definition 2.1. These seminorms hold for any measure preserving system.

Definition 4.4. Suppose (Y, \mathcal{Y}, ν, U) is a measure preserving system, and $f \in L^\infty(\nu)$. We define seminorms $\|\langle \cdot \rangle\|_2$ and $\|\langle \cdot \rangle\|_3$ so that

$$\|\langle f \rangle\|_2^4 = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \left| \int f \cdot f \circ U^h d\nu \right|^2,$$

and

$$\|\langle f \rangle\|_3^8 = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|\langle f \cdot f \circ U^h \rangle\|_2^4$$

Certainly, if U is ergodic, then $\|\langle f \rangle\|_k = \|f\|_k$ for $k = 2, 3$. We can easily verify that $\|\langle \cdot \rangle\|_2$ and $\|\langle \cdot \rangle\|_3$ are indeed seminorms. For example, $\|\langle \cdot \rangle\|_2$ is a positive semidefinite function, since

$$\begin{aligned} \|\langle f \rangle\|_2^4 &= \lim_{H \rightarrow \infty} \frac{1}{H} \left| \int f \cdot f \circ U^h d\nu \right|^2 \\ &= \lim_{H \rightarrow \infty} \frac{1}{H} \int (f \cdot f \circ U^h)(x) d\nu(x) \int (f \cdot f \circ U^h)(y) d\nu(y) \\ &= \iint f(x)f(y) \mathbb{E}(f \otimes f | \mathcal{I}^2)(x, y) d\nu \otimes \nu(x, y) \\ &= \iint \mathbb{E}(f \otimes f | \mathcal{I}^2)^2(x, y) d\nu \otimes \nu(x, y) \geq 0, \end{aligned}$$

where \mathcal{I}^2 is the sigma-algebra generated by $U \times U$ -invariant sets.

Lemma 4.5. Suppose (Y, \mathcal{Y}, ν, U) be a measure preserving system, and $f \in L^\infty(\nu)$. Then

$$\int \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N f(U^n y) \right|^2 d\nu \leq C \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \left| \int f \cdot f \circ U^h d\nu \right| \leq C \|\langle f \rangle\|_2^2$$

for ν -a.e. $y \in Y$.

Proof. We denote $F_h(x) = f(x)f \circ U^h(x)$. From van der Corput's lemma, we obtain, for $0 \leq H \leq N-1$,

$$\left| \frac{1}{N} \sum_{n=1}^N f(U^n y) \right|^2 \leq \frac{C}{H} + \frac{4}{(H+1)^2} \sum_{h=1}^H (H+1-h) \left(\frac{1}{N} \sum_{n=0}^{N-h} F_h(U^n y) \right).$$

Birkhoff's pointwise ergodic theorem tells us that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-h} F_h(U^n y) = \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=0}^N F_h(U^n y) - \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=N-h+1}^N F_h(U^n y) \right) = \mathbb{E}(f \cdot f \circ U^h | \mathcal{I})(y)$$

for ν -a.e. $y \in Y$, where \mathcal{I} is the sigma-algebra generated by U -invariant sets. And because

$$\int \mathbb{E}(f \cdot f \circ U^h | \mathcal{I}) d\nu = \int f \cdot f \circ U^h d\nu,$$

we have

$$\begin{aligned} \int \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N f(U^n y) \right|^2 d\nu &\leq \frac{C}{H} + \frac{4}{(H+1)^2} \sum_{h=1}^H (H+1-h) \left(\int f \cdot f \circ U^h d\nu \right) \\ &\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \left| \int f \cdot f \circ U^h d\nu \right| \\ &\leq \frac{C}{H} + C \left(\frac{1}{H} \sum_{h=1}^H \left| \int f \cdot f \circ U^h d\nu \right|^2 \right)^{1/2} \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality. If we let $N \rightarrow \infty$, we can also let $H \rightarrow \infty$ to obtain

$$\int \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N f(U^n y) \right|^2 d\nu \leq C \left(\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \left| \int f \cdot f \circ U^h d\nu \right|^2 \right)^{1/2} = C \|\langle f \rangle\|_2^2$$

□

Lemma 4.6. Suppose (X, \mathcal{F}, μ, T) be an ergodic dynamical system, and $f_1, f_2 \in L^\infty(\mu)$. Then for any integers a and b ,

$$\int \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}y) \right|^2 d\mu \otimes \mu(x, y) \leq C|a| \|f_2\|_\infty^2 \|f_1\|_2^2$$

for μ -a.e. $x, y \in X$.

Proof. We denote $F_{1,h}(x) = f_1(x)f_1 \circ T^{ah}(x)$, and $F_{2,h}(x) = f_2(x)f_2 \circ T^{bh}(x)$. If $U = T^a \times T^b$, then $(X^2, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu, U)$ is a measure preserving system. Hence, we can apply lemma 4.5 to obtain, for any $1 \leq H \leq N-1$,

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N f_1 \otimes f_2(U^n(x, y)) \right|^2 d\mu \otimes \mu(x, y) \\ & \leq \frac{C}{H} \sum_{h=1}^H \left| \iint F_{1,h}(x) F_{2,h}(y) d\mu \otimes \mu(x, y) \right| \\ & = \frac{C}{H} \sum_{h=1}^H \left| \int F_{1,h}(x) d\mu(x) \right| \left| \int F_{2,h}(y) d\mu(y) \right| \\ & \leq C \|f_2\|_\infty^2 \left(\frac{1}{H} \sum_{h=1}^H \left| \int f \cdot f \circ T^{ah} d\mu \right|^2 \right)^{1/2}, \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality. As we let $H \rightarrow \infty$, we obtain the desired result by lemma 4.2. □

Lemma 4.7. Suppose (X, \mathcal{F}, μ, T) is an ergodic system, and $f_1, f_2 \in L^\infty(\mu)$. Then for any integers a and b ,

$$\int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}y) e^{2\pi i n t} \right|^2 d\mu \otimes \mu(x, y) \leq C|a|^{3/2} \|f_2\|_\infty^4 \|f_1\|_3^4$$

for μ -a.e. $x, y \in X$.

Proof. We denote $F_{1,h}(x) = f_1(x)f_1 \circ T^{ah}(x)$, and $F_{2,h}(x) = f_2(x)f_2 \circ T^{bh}(x)$. By van der Corput's lemma, we obtain

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}y) e^{2\pi i n t} \right|^2 \\ & \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \left| \frac{1}{N} \sum_{n=1}^N F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y) \right| \end{aligned}$$

Again, if we set $U = T^a \times T^b$, then $(X^2, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu, U)$ is a measure preserving system. Hence, Birkhoff's pointwise ergodic theorem asserts that the average $\frac{1}{N} \sum_{n=1}^N F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y)$ converges $\mu \otimes \mu$ -a.e. Hence,

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}y) e^{2\pi i n t} \right|^2 d\mu \otimes \mu(x, y) \\ & \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \int \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y) \right| d\mu \otimes \mu(x, y) \\ & \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \left(\int \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y) \right|^2 d\mu \otimes \mu(x, y) \right)^{1/2} \end{aligned}$$

By lemma 4.6, we know that

$$\int \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N F_{1,h}(T^{an}x) F_{2,h}(T^{bn}y) \right|^2 d\mu \otimes \mu(x, y) \leq C|a| \|F_{2,h}\|_{\infty}^2 \|F_{1,h}\|_2^2 = C|a| \|f_2\|_{\infty}^4 \|F_{1,h}\|_2^2.$$

Hence,

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}y) e^{2\pi i n t} \right|^2 d\mu \\ & \leq \frac{C|a| \|f_2\|_{\infty}^4}{H} \sum_{h=1}^H \|f_1 \cdot f_1 \circ T^{ah}\|_2^2 \leq C|a| \|f_2\|_{\infty}^4 \left(\frac{1}{H} \sum_{h=1}^H \|f_1 \cdot f_1 \circ T^{ah}\|_2^4 \right)^{1/2}. \end{aligned}$$

Let $H \rightarrow \infty$, and apply lemma 4.2 to obtain the desired result. \square

Now we are ready to prove the main uniform Wiener Wintner double recurrence theorem.

Theorem 4.8. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system, and $f_1, f_2 \in L^{\infty}(X)$, and $\|f_2\|_{\infty} = 1$. If $f_1 \in \mathcal{Z}_2^{\perp}$, then*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t} \right| = 0$$

for μ -a.e. $x \in X$, and for any pair of integers a and b .

Proof. We denote $F_{1,h}(x) = f_1(x) f_1 \circ T^{ah}(x)$, and $F_{2,h}(x) = f_2(x) f_2 \circ T^{bh}(x)$. By the van der Corput's lemma, we have, for any $0 < H < N$,

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t} \right| \\ & \leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^H \left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right| \\ & \leq \frac{C}{H} + \frac{C}{H} \left(\sum_{h=1}^H \left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right|^2 \right)^{1/2}, \end{aligned}$$

where the second inequality is the consequence of the Cauchy-Schwarz inequality. Note that we can also

apply the van der Corput's lemma on the average $\left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right|^2$ to obtain the following bound for $0 < K < N$.

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right|^2 \\ & \leq \frac{C'}{K} + \frac{C'}{K} \sum_{k=1}^K \left(\frac{1}{N} \sum_{n=1}^{N-k-1} (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x) (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x) \right). \end{aligned}$$

Note that the average

$$\frac{1}{N} \sum_{n=1}^{N-k-1} (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x) \cdot (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x)$$

converges as $N \rightarrow \infty$ by Bourgain's a.e. double recurrence theorem in [7]. Therefore,

$$\begin{aligned}
& \int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t} \right| d\mu(x) \\
& \leq \frac{C}{H} + \frac{C}{H} \int \left(\sum_{h=1}^H \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right|^2 \right)^{1/2} d\mu(x) \\
& \leq \frac{C}{H} + \frac{C}{H} \left(\sum_{h=1}^H \int \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-h-1} F_{1,h}(T^{an}x) F_{2,h}(T^{bn}x) \right|^2 d\mu(x) \right)^{1/2} \\
& \leq \frac{C}{H} + \frac{C}{H} \left(\sum_{h=1}^H \left(\frac{C'}{K} + \frac{C'}{K} \sum_{k=1}^K \left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-k-1} \int (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x) (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x) d\mu(x) \right) \right) \right)^{1/2},
\end{aligned}$$

where the second inequality follows from Hölder's inequality. Since T^a is a measure preserving transformation, we obtain

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-k-1} \int (F_{1,h} \cdot F_{1,h} \circ T^{ak})(T^{an}x) (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{bn}x) d\mu(x) \\
& = \int (F_{1,h} \cdot F_{1,h} \circ T^{ak})(x) \left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-k-1} (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{(b-a)n}x) \right) d\mu(x).
\end{aligned}$$

Observe that Birkhoff's pointwise ergodic theorem tells us that

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N-k-1} (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{(b-a)n}x) \\
& = \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=1}^{N-k-1} (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{(b-a)n}x) - \frac{1}{N} \sum_{n=1}^{N-k-1} (F_{2,h} \cdot F_{2,h} \circ T^{bk})(T^{(b-a)n}x) \right) \\
& = \mathbb{E}(F_{2,h} \cdot F_{2,h} \circ T^{bk} | \mathcal{I}_{b-a})(x),
\end{aligned}$$

for μ -a.e. $x \in X$, where \mathcal{I}_{b-a} is the σ -algebra generated by T^{b-a} -invariant sets. By lemma 4.1, there exists a positive integer l_{b-a} and partition $A_1, \dots, A_{l_{b-a}}$ of X such that

$$\mathbb{E}(F_{2,h} \cdot F_{2,h} \circ T^{bk} | \mathcal{I}_{b-a})(x) = \int (F_{2,h} \cdot F_{2,h} \circ T^{bk})(y) K_{b-a}(x, y) d\mu(y),$$

where $K_{b-a}(x, y) = l_{b-a} \sum_{i=1}^{l_{b-a}} \mathbb{1}_{A_i}(x) \mathbb{1}_{A_i}(y)$. Note that

$$\begin{aligned}
& \iint (F_{1,h}(x) \cdot F_{1,h} \circ T^{ak})(x) (F_{2,h} \cdot F_{2,h} \circ T^{bk})(y) K(x, y) d\mu(x) d\mu(y) \\
& = \iint f_1 \otimes f_2(x, y) K(x, y) f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y)) d\mu \otimes \mu(x, y),
\end{aligned}$$

where $U = T^a \times T^b$ is a measure preserving transformation on X^2 . Let $H = K$. Note that, on the system $(X^2, \mu \otimes \mu, U)$, lemma 5 from [3] tells us that we have

$$\begin{aligned}
& \frac{1}{H^2} \left| \sum_{h,k=0}^{H-1} f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y)) \right| \\
(8) \quad & \leq \limsup_{H \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{H} \sum_{h=1}^H f_1 \otimes f_2(U^h(x, y)) e^{2\pi i h t} \right|.
\end{aligned}$$

By lemma 4.7, we know that

$$\begin{aligned} & \int \limsup_{H \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{H} \sum_{h=1}^H f_1 \otimes f_2(U^h(x, y)) e^{2\pi i h t} \right| d\mu \otimes \mu(x, y) \\ &= \int \limsup_{H \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{H} \sum_{h=1}^H f_1(T^{ah}x) f_2(T^{bh}y) e^{2\pi i h t} \right| d\mu \otimes \mu(x, y) \\ &\leq C|a| \|f_1\|_3^4 = 0. \end{aligned}$$

Hence, by Lebesgue's dominated convergence theorem,

$$\begin{aligned} & \int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}y) e^{2\pi i n t} \right| d\mu \otimes \mu(x, y) \\ &\leq \iint f_1 \otimes f_2(x, y) K(x, y) \lim_{H \rightarrow \infty} \frac{1}{H^2} \sum_{h,k=0}^{H-1} \left(f_1 \otimes f_2(U^h(x, y)) f_1 \otimes f_2(U^k(x, y)) f_1 \otimes f_2(U^{h+k}(x, y)) \right) d\mu \otimes \mu(x, y) \\ &= 0. \end{aligned}$$

Therefore, $\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}y) e^{2\pi i n t} \right| = 0$, so the right hand side of the inequality (8) equals zero. This concludes the proof. \square

5. CASE WHEN $f_1 \in \mathcal{A}_2^\perp$, $a = 1$, $b = 2$

In this section, we will prove the following pointwise estimate, which we can use to prove the uniform Wiener Wintner double recurrence theorem for the case $f_1 \in \mathcal{A}_2^\perp$. We will prove this case for $a = 1$ and $b = 2$.

Theorem 5.1. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system. Then there exist a universal constant C such that*

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right| \leq C N_3(f_1)^2$$

for μ -a.e. $x \in X$. In particular, if $f_1 \in \mathcal{A}_2^\perp$, then

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right| = 0$$

for μ -a.e. $x \in X$.

Remark: While this is a special case of the result in the preceding section (since $\mathcal{A}_2^\perp \subset \mathcal{Z}_2^\perp$), we wish to emphasize here that we can bound the double recurrences averages using the seminorm $N_2(\cdot)$ without taking the integral of the norm of the averages. This was not the case when we used the Host-Kra seminorm $\|\cdot\|_3$, where we obtained the norm bound

$$\int \limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right| d\mu \leq C \|f_1\|_3^2$$

Proof. We start the proof of Theorem 5.1 by applying the van der Corput's lemma. Let $a_n = f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t}$. Then,

$$\begin{aligned} \sup_{t \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^N f_1(T^n x) f_2(T^{2n} x) e^{2\pi i n t} \right|^2 &\leq \frac{1}{N(H+1)^2} \sum_{n=1}^N |a_n|^2 \\ &\quad + \frac{2}{N(H+1)^2} \sum_{h=1}^H (H+1-h) \Re \left(\frac{e^{-2\pi i h t}}{N} \sum_{n=1}^{N-h-1} a_n a_{n+h} \right) \\ &\leq \frac{1}{N(H+1)^2} \sum_{n=1}^N |a_n|^2 + \frac{2}{N(H+1)^2} \sum_{h=1}^H (H+1-h) \left| \frac{1}{N} \sum_{n=1}^{N-h-1} a_n a_{n+h} \right| \end{aligned}$$

Since $\|f_1\|_\infty, \|f_2\|_\infty < \infty$, we know that $\frac{1}{N(H+1)^2} \sum_{n=1}^N |a_n|^2$ converges to 0 as $N \rightarrow \infty$ and $H \rightarrow \infty$, since $|a_n| \leq \|f_1\|_\infty \|f_2\|_\infty$. Hence, our main task is to show that

$$(9) \quad \limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N (f_1 \cdot f_1 \circ T^h)(T^n x) (f_2 \cdot f_2 \circ T^{2h})(T^{2n} x) \right| = 0$$

We will first prove the following lemma:

Lemma 5.2. *Suppose $F_1, F_2 \in L^\infty(X)$. If $F_1 \in \mathcal{A}_1^\perp$, then for μ -a.e. $x \in X$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) = 0.$$

Thus, \mathcal{A}_1 is a pointwise characteristic factor of this average, i.e.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbb{E}(F_1 | \mathcal{A}_1)(T^n x) \mathbb{E}(F_2 | \mathcal{A}_1)(T^{2n} x).$$

Proof. Since $\left| \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) \right|$ is non-negative, we can prove this lemma by showing

$$\lim_{N \rightarrow \infty} \int \left| \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) \right| d\mu(x) = \int \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) \right| d\mu(x) = 0$$

where the first equality holds by Bourgain's double recurrence theorem and Lebesgue's dominated convergence theorem. Note that Cauchy-Schwarz inequality asserts that

$$\int \left| \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) \right| d\mu(x) \leq \left(\int \left| \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) \right|^2 d\mu(x) \right)^{1/2}.$$

We will proceed by using the van der Corput's lemma. Observe that

$$\begin{aligned} &\int \left| \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) \right|^2 d\mu(x) \\ &\leq \frac{N+H}{N^2(H+1)} \sum_{n=0}^{N-1} \int |F_1(T^n x) F_2(T^{2n} x)|^2 d\mu(x) \\ &\quad + \frac{2(N+H)}{N(H+1)^2} \sum_{h=1}^H (H+1-h) \left(\frac{1}{N} \sum_{n=1}^{N-h-1} \int (F_1 \cdot F_1 \circ T^h)(T^n x) (F_2 \cdot F_2 \circ T^{2h})(T^{2n} x) d\mu(x) \right) \end{aligned}$$

By letting $N \rightarrow \infty$, we obtain

$$\lim_{N \rightarrow \infty} \int \left| \frac{1}{N} \sum_{n=1}^N F_1(T^n x) F_2(T^{2n} x) \right|^2 d\mu(x) \leq \frac{\|F_2\|_\infty^2}{H} \sum_{h=1}^H \left| \int (F_1 \cdot F_1 \circ T^h)(x) d\mu(x) \right|,$$

since T is ergodic and hence measure preserving. Note that

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \left| \int (F_1 \cdot F_1 \circ T^h)(x) d\mu(x) \right| = \lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H |\hat{\sigma}_{F_1}(h)| \leq \left(\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H |\hat{\sigma}_{F_1}(h)|^2 \right)^{1/2},$$

and because $F_1 \in \mathcal{A}_1^\perp$, the spectral measure σ_{F_1} is continuous, so the Wiener's theorem implies the right hand side of above limit equals 0. \square

We will conclude the theorem by applying the following estimate on (9).

Lemma 5.3. *Let (X, \mathcal{F}, μ, T) be an ergodic system. Let $f_1, f_2 \in L^\infty(X)$. Then there exists $C > 0$ such that*

$$\limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N (f_1 \cdot f_1 \circ T^h)(T^n x) (f_2 \cdot f_2 \circ T^{2h})(T^{2n} x) \right| \leq CN_2(f_1)^2$$

for μ -a.e. $x \in X$.

Proof. Set $F_1 = f_1 \cdot f_1 \circ T^h$, and $F_2 = f_2 \cdot f_2 \circ T^{2h}$. Denote

$$P_N(F_1, F_2) = \frac{1}{N} \sum_{n=1}^N \mathbb{E}(F_1 | \mathcal{A}_1) \circ T^n \mathbb{E}(F_2 | \mathcal{A}_1) \circ T^{2n}.$$

Let $\{e_j\}$ be an eigenbasis of \mathcal{A}_1 , where λ_j is a corresponding eigenvalue of e_j . Then we would have

$$\mathbb{E}(F_1 | \mathcal{A}_1) \circ T^n = \sum_{j=0}^{\infty} \left(\int F_1 e_j d\mu \right) \lambda_j^n e_j \text{ and } \mathbb{E}(F_2 | \mathcal{A}_1) \circ T^n = \sum_{l=0}^{\infty} \left(\int F_2 e_l d\mu \right) \lambda_l^n e_l$$

in L^2 -norm. Hence,

$$\lim_{N \rightarrow \infty} P_N(F_1, F_2) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \left(\int F_1 \bar{e}_j d\mu \right) \left(\int F_2 \bar{e}_l d\mu \right) \lambda_j^n \lambda_l^{2n} e_j e_l$$

Note that for each j and l ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda_j^n \lambda_l^{2n} = \begin{cases} 1 & \text{if } \lambda_j = \bar{\lambda}_l^2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if we denote $R = \{(j, l_j) \in \mathbb{N}^2 : \lambda_j = \bar{\lambda}_{l_j}^2\}$, then

$$\lim_{N \rightarrow \infty} P_N(F_1, F_2) = \sum_{(j, l_j) \in R} \left(\int F_1 \bar{e}_j d\mu \right) \left(\int F_2 \bar{e}_{l_j} d\mu \right) e_j e_{l_j}$$

Note that the sequence

$$B_J = \left(\sum_{(j, l_j) \in R, j \leq J} \left(\int F_1 \bar{e}_j d\mu \right) \left(\int F_2 \bar{e}_{l_j} d\mu \right) e_j e_{l_j} \right)_J$$

converges to $\lim_{N \rightarrow \infty} P_N(F_1, F_2)$ in norm as $J \rightarrow \infty$. Therefore, there exists a subsequence $(B_{J_k})_k$ that converges to $\lim_{N \rightarrow \infty} P_N(F_1, F_2)(x)$ for μ -a.e. $x \in X$. Thus,

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbb{E}(F_1 | \mathcal{A}_1)(T^n x) \mathbb{E}(F_2 | \mathcal{A}_1)(T^{2n} x) \\
&= \lim_{k \rightarrow \infty} \sum_{(j, l_j) \in R, j \leq J_k} \left(\int F_1 \bar{e}_j d\mu \right) \left(\int F_2 \bar{e}_{l_j} d\mu \right) e_j(x) e_{l_j}(x) \\
&\leq \lim_{k \rightarrow \infty} \left(\sum_{(j, l_j) \in R, j \leq J_k} \left| \int F_1 \bar{e}_j d\mu \right|^2 \right)^{1/2} \left(\sum_{(j, l_j) \in R, j \leq J_k} \left| \int F_2 \bar{e}_{l_j} d\mu \right|^2 \right)^{1/2} \\
&\leq \left(\sum_{j=1}^{\infty} \left| \int F_1 \bar{e}_j d\mu \right|^2 \right)^{1/2} \left(\sum_{l=1}^{\infty} \left| \int F_2 \bar{e}_l d\mu \right|^2 \right)^{1/2} \\
&= \|\mathbb{E}(F_1 | \mathcal{A}_1)\|_2 \|\mathbb{E}(F_2 | \mathcal{A}_1)\|_2 \\
&\leq C \|\mathbb{E}(F_1 | \mathcal{A}_1)\|_2,
\end{aligned}$$

where $C \geq \|f_2\|_{\infty}^2$. Therefore,

$$\begin{aligned}
& \limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left(f_1 \cdot f_1 \circ T^h \right) (T^n x) \left(f_2 \cdot f_2 \circ T^{2h} \right) (T^{2n} x) \\
&\leq C \limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|\mathbb{E}(f_1 \cdot f_1 \circ T^h | \mathcal{A}_1)\|_2 \\
&\leq C \left(\limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \|\mathbb{E}(f_1 \cdot f_1 \circ T^h | \mathcal{A}_1)\|_2^2 \right)^{1/2} = C N_2(f_1)^2,
\end{aligned}$$

where the second inequality holds by the Cauchy-Schwarz inequality. \square

To conclude the proof of theorem 5.1, just note that when $f_1 \in \mathcal{A}_2^{\perp}$, then $N_2(f)^2 = 0$, so by lemma 5.3, the limit in (9) goes to 0. \square

6. CASE WHEN BOTH $f_1, f_2 \in \mathcal{Z}_2$

Let (X, \mathcal{F}, μ, T) be an ergodic system. Recall that X is called a k -step nilsystem if X is a homogeneous space of a k -step nilpotent Lie group G (such a manifold is called a nilmanifold). Let Λ be a discrete cocompact subgroup of G such that $X = G/\Lambda$. The outline of the proof of the following theorem is given in [15].

Theorem 6.1 (Host, Kra [15]). *If X is a Conze-Lesigne system, then it is the inverse limit of a sequence of 2-step nilsystems.*

In the outline of the proof, X is reduced to the case where X is a group extension of the Kronecker factor Z_1 and torus U , with cocycle $\rho : Z_1 \rightarrow U$. A group G is defined to be a family of transformations of $X = Z_1 \times U$, where U is a finite dimensional torus and Z_1 is the Kronecker factor of X that has the structure of compact abelian Lie group. If $g \in G$, $(z, u) \in X$, then

$$g(z, u) = (sz, uf(z))$$

where s and f satisfy the Conze-Lesigne equation

$$\rho(sz)\rho(z)^{-1} = f(Rz)f(z)^{-1}c$$

for some constant $c \in U$. It can be easily verified that G is a 2-step nilpotent group, and T corresponds to $(\beta, \rho) \in G$, where $\beta \in Z_1$ such that if $\pi_1 : Z_2 \rightarrow Z_1$ is a factor map, then $\pi_1(T_1 x) = \beta \pi_1(x)$. Furthermore, if G is given a topology of convergence in probability, we can show that G is a Lie group.

The outline of the proof given in [15] concludes by stating that G acts on X transitively, and X can be identified with the nilmanifold G/Λ , where Λ is a stabilizer group of a point $x_0 \in X$ (hence it is a discrete cocompact subgroup of G). Furthermore, μ is a Haar measure on X , and T is a translation by the element $(\beta, \rho) \in G$. Hence, T acts on X by translation. We will use this fact to prove the convergence of the double recurrence Wiener Wintner average for the case when $f_1, f_2 \in \mathcal{Z}_2$.

The following convergence result of Leibman will be used. We say $\{g(n)\}_{n \in \mathbb{Z}}$ is a *polynomial sequence* if $g(n) = a_1^{p_1(n)} \cdots a_m^{p_m(n)}$, where $a_1, \dots, a_m \in G$, and p_1, \dots, p_m are polynomials taking on integer value on the integers.

Theorem 6.2 (Leibman [16]). *Let $X = G/\Lambda$ be a nilmanifold and $\{g(n)\}_{n \in \mathbb{Z}}$ be a polynomial sequence in G . Then for any $x \in X$ and continuous function F on X , the average*

$$\frac{1}{N} \sum_{n=1}^N F(g(n)x)$$

converges.

Theorem 6.3. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system. Suppose $f_1, f_2 \in \mathcal{Z}_2$ are both continuous functions on X . Then the average*

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t}$$

converges off of a single null-set that is independent of t .

Proof. In this proof, we will consider two cases: The case when t is rational, and the case when t is irrational.

Case I: When t is rational. Fix $t \in \mathbb{Q}$. Let S_t be a rotation on \mathbb{T} by $e^{2\pi i t}$. Let $(X \times \mathbb{T}, \mu \otimes m, U)$ be a measure preserving system, where m is the Lebesgue measure on \mathbb{T} , and $U = T \otimes S_t$. Define $F_1(x, y) = f_1(x) e^{2\pi i \alpha_1 y}$, and $F_2(x, y) = f_2(x) e^{2\pi i \alpha_2 y}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 a + \alpha_2 b = 1$. Then

$$(10) \quad \frac{1}{N} \sum_{n=1}^N F_1(U^{an}(x, y)) F_2(U^{bn}(x, y)) = \frac{e^{2\pi i y}}{N} \sum_{n=1}^N f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t}$$

Note that the average on the left hand side of (10) converges $\mu \otimes m$ -a.e. as $N \rightarrow \infty$ by Bourgain's double recurrence theorem [7]. So there exists a set of full measure $V_t \subset X \times \mathbb{T}$ such that the average in (10) converges for all $(x, y) \in V_t$. If $V = \bigcup_{t \in \mathbb{Q}} V_t$, then V is a set of full measures such that the average on (10) converges for all $(x, y) \in V$ for all $t \in \mathbb{Q}$. This implies that the claim holds for μ -a.e. $x \in X$ when $t \in \mathbb{Q}$.

Case II: When t is irrational. Without loss of generality, we let $X = \mathbb{Z}_2$, the Conze-Lesigne system. Let $\beta \in \mathbb{Z}_1$ is an element such that for any $(z, u) \in \mathbb{Z}_1 \times U = X$, $T(z, u) = (\beta z, u\rho(z))$. In other words, T acts on \mathbb{Z}_1 as a rotation by β (here, we let \mathbb{Z}_1 be a multiplicative abelian group). Then note that $B = \langle \beta \rangle$, the cyclic subgroup generated by β , is dense in the Kronecker factor \mathbb{Z}_1 . Define a character $\phi_t : B \rightarrow \mathbb{T}$ such that $\phi_t(\beta) = e^{2\pi i t}$. Such group homomorphism exists since t is irrational, and $\langle e^{2\pi i t} \rangle$ generates a dense cyclic subgroup in \mathbb{T} .

We claim that there exists a multiplicative character $\bar{\phi}_t : \mathbb{Z}_1 \rightarrow \mathbb{T}$ such that $\bar{\phi}_t|_B = \phi_t$. Since B is dense in \mathbb{Z}_1 , for any $z \in \mathbb{Z}_1$, there exists a sequence $(\beta^{n_k})_k$ such that $\lim_{k \rightarrow \infty} \beta^{n_k} = z$. So we define

$$\bar{\phi}_t(z) = \lim_{k \rightarrow \infty} \phi_t(\beta)^{n_k}.$$

We must show that this limit converges, and the function $\bar{\phi}_t$ is well-defined. Note that \mathbb{T} is compact, so there exists a converging subsequence $(\phi_t(\beta)^{n_{k_l}}) \in \mathbb{T}$ such that $\lim_{l \rightarrow \infty} \phi_t(\beta)^{n_{k_l}} = \gamma$ for some $\gamma \in \mathbb{T}$. We will show that $\lim_{k \rightarrow \infty} \phi_t(\beta)^{n_k} = \gamma$. Assuming on the contrary, suppose that there exists a subsequence $(\phi_t(\beta)^{n_{k_m}})_m$ such that $|\phi_t(\beta)^{n_{k_m}} - \gamma| > \epsilon$ for all $m \in \mathbb{N}$. This implies that, for sufficiently large l , we have $|\phi_t(\beta)^{n_{k_m}} - \phi_t(\beta)^{n_{k_l}}| > \epsilon/2$. This however contradicts the continuity of ϕ_t , since $d_{\mathbb{Z}_1}(\beta^{n_{k_l}}, \beta^{n_{k_m}}) \rightarrow 0$ as $l, m \rightarrow \infty$, since both $\beta^{n_{k_l}}$ and $\beta^{n_{k_m}}$ converges to the same limit z . This proves that $\bar{\phi}_t$ is well-defined for all $z \in \mathbb{Z}_1$. The fact that $\bar{\phi}_t$ is a multiplicative character is obvious from the way $\bar{\phi}_t$ is defined in terms of ϕ_t .

We define a continuous function $f_t := \bar{\phi}_t \circ \pi_1$, where $\pi_1 : Z_2 \rightarrow Z_1$ is a factor map. We note that

$$f_t(T^n x) = \bar{\phi}_t(\pi_1(T^n x)) = \bar{\phi}_t(\pi_1(x)\beta^n) = f_t(x)\phi_t(\beta)^n = f_t(x)e^{2\pi i n t}.$$

Therefore,

$$\frac{1}{N} \sum_{n=1}^N f_1(T^{an}x)f_2(T^{bn}x)f_t(T^n x) = \frac{f_t(x)}{N} \sum_{n=1}^N f_1(T^{an}x)f_2(T^{bn}x)e^{2\pi i n t}.$$

To show the convergence of this average, let $F(x_1, x_2, x_3) = f_1(x_1)f_2(x_2)f_t(x_3)$ be a function on $X^3 = G^3/\Lambda^3$. Let $T_1 = T \times \text{Id} \times \text{Id}$, $T_2 = \text{Id} \times T \times \text{Id}$, and $T_3 = \text{Id} \times \text{Id} \times T$. Note that an action of T_1 on X^3 corresponds to $g_1 = ((\beta, \rho), e, e) \in G^3$ (where e is the identity element of G), and similarly, T_2 corresponds to $g_2(e, (\beta, \rho), e) \in G^3$, and T_3 corresponds to $g_3(e, e, (\beta, \rho)) \in G^3$. Thus,

$$g(n) = g_1^{an} g_2^{bn} g_3^n$$

is a polynomial sequence. Furthermore, if $\vec{x} = (x, x, x) \in X^3$, then

$$\frac{1}{N} \sum_{n=1}^N F(g(n)\vec{x}) = \frac{1}{N} \sum_{n=1}^N f_1(T^{an}x)f_2(T^{bn}x)f_t(T^n x)$$

converges by theorem 6.2. □

Remark: The first and the third authors are currently preparing the extension of theorem 2.3 to show that the sequence $u_n = f_1(T^{an}x)f_2(T^{bn}x)$ is a good universal weight for the pointwise ergodic theorem [4].

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